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Curvature Calculations with GEOCALC

A. Moussiaux[1] and Ph. Tombal[1]

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A new method for calculating the curvature tensor has been recently proposed by D. Hestenes. This method is a particular application of geometric calculus, which has been implemented in an algebraic programming language on the form of a package called GEOCALC. We show how to apply this package to the Schwarzchild case and we discuss the different results.

1. INTRODUCTION

In a recent paper Hestenes (1986) applied a new method for calculating the curvature tensor in the Schwarzchild case. The method is an application of geometric calculus and Clifford algebra (Hestenes and Sobcszyk, 1984). The fundamental concepts of geometric calculus are multivectors and geometric products, from which other products, such as the dot product, outer product, and scalar product, are particular cases.

Most computations in general relativity are usually performed in an algebraic programming language,[2] and are based on the same theoretical background.

Geometric calculus is a nonconventional method having a larger range of applications than the tetrad formalism or exterior calculus, for example. Implementation of geometric calculus in an algebraic programming language provides a powerful tool for theoretical investigations. In a previous paper (Tombal and Moussiaux, 1986) we explained how the fundamental operations of geometric calculus have been implemented in MACSYMA, creating a package called GEOCALC. Adding a derivative to GEOCALC allowed us to create a new package devoted to curvature

[1]Département de Physique, Facultés Universitaires, 5000 Namur, Belgium.

[2]For example, REDUCE (User's Manual, Anthony C. Hearn), SHEEP (I. Frick, Institute of Theoretical Physics, University of Stockholm), MACSYMA (Reference Manual, Symbolics Inc., MIT, Cambridge, Massachusetts), EXCALC (User's Manual, Eberhard Schrufer, 1986).

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calculations using the general method of "fiducial frame" developed by Hestenes and Sobczyk (1984). This package revealed some computational errors in Hestenes (1986), showing, if it is still necessary, the advantages of algebraic computation compared with a human one.

In Section 2 we recall briefly the principles of the "fiducial frame" and how it is adapted to our own computation. In Section 3 the generalized gradient operator is introduced. Section 4 recalls how to compute the curvature, Riemann, and Ricci tensors following Hestenes (1986). Section 5 explains how the quantities of Section 4 are translated in GEOCALC. In Section 6 we compare our results with those of Hestenes (1986).

2. THE "FIDUCIAL FRAME" METHOD

According to Hestenes (1986), the fiducial frame $\{\gamma_i\}$ is related to the coordinate frame $\{g_j\}$ by a fiducial tensor

$$g_j = h^i_j \gamma_i \tag{1}$$

The components of the metric tensor are given by

$$g_{ij} = g_i \cdot g_j \tag{2}$$

The fiducial frame is orthogonal; then

$$\gamma_i \cdot \gamma_j = \eta_i \delta_{ij} \tag{3}$$

where η_i is the signature indicator.

Relation (3) shows that for $i = j$ the dot product of the two vectors γ_i and γ_j gives the scalar η_i. Following Hestenes, the geometric product of γ_i and γ_j is given by

$$\gamma_i \gamma_j = \gamma_i \cdot \gamma_j + \gamma_i \wedge \gamma_j \tag{4}$$

If the vectors γ_i and γ_j are chosen such that $\gamma_i \wedge \gamma_j = 0$, comparison of (3) and (4) forces us to choose a Clifford algebra essentially determined by the signature indicator η_i.

The γ_i are chosen to be four 1-vectors expressed, as explained in Tombal and Moussiaux (1986), in terms of blades in the following way:

$$\gamma_i = @(1, [i]) \tag{5}$$

so that the relations (3) are automatically satisfied.

A reciprocal frame γ^j is defined such that

$$\gamma_i \cdot \gamma^j = \delta^j_i \tag{6}$$

These vectors are represented by

$$\gamma^j = @(\eta_j, [j]) \tag{7}$$

Here the γ_i and γ^j are constant vectors.

In Hestenes' notations a diagonal metric in a Lorentz tetrad can always be written

$$ds^2 = \sum_{i=0}^{3} \eta_i (h_i)^2 (dx^i)^2 \tag{8}$$

where the h_i are function of the coordinates (x^0, x^1, x^2, x^3). In this case the fiducial tensor is very simple, so that the coordinate frame is given by

$$g_i = @(h_i, [i]) \tag{9}$$

and the corresponding reciprocal coordinate frame is

$$g^j = @(\eta_j h_j^{-1}, [j]) \tag{10}$$

It is clear that (10) and (11) satisfy the relations (2) defining the metric g_{ij} as well as the relation

$$g_i \cdot g^j = \delta_i^j \tag{11}$$

3. THE DIFFERENTIAL OPERATORS

With a set of scalar coordinates (x^0, x^1, x^2, x^3) we can always construct a 1-vector differential operator

$$\mathbf{d} = (\partial_0, \partial_1, \partial_2, \partial_3) \tag{12}$$

This operator behaves like any other 1-vector with respect to the operations defined in geometric calculus. But of course **d** must have also the usual properties of a differential operator.

If the metric is given by (8), the relevant differential operator is

$$\square = (\gamma^i / h_i)\, \partial_i = g^i\, \partial_i \tag{13}$$

We easily check that the fiducial frame is related to the coordinates by

$$\gamma_i = \eta_i h_i \square x^i \tag{14}$$

where there is no summation on the index i.

Equation (13) shows that $\square$ is nothing else than a generalization of the gradient operator. This operator has the same vectorial and differential properties as **d**.

4. THE CURVATURE COMPUTATION

Following Hestenes (1986), the curvature tensor evaluated on the bivector $g_i \wedge g_j$ is given by

$$\omega_{ij} = d_i \omega_j - d_j \omega_i + \omega_i \times \omega_j = R(g_i \wedge g_j) \tag{15}$$

where the fiducial derivative $d_i\omega_j$ must be computed with

$$d_i\omega_j = \partial_i(h_k^{-1}\,\partial_k h_j)\gamma_j \wedge \gamma^k \tag{16}$$

and where the connection bivectors ω_i are defined by

$$\omega_i = \gamma_i \wedge \Box h_i \tag{17}$$

In (15) the symbol $\times$ stands for the cross product defined by $A\times B = (AB-BA)/2$. In a natural base the covariant components of the Riemann tensor are related to the curvature tensor R by

$$R_{ijkl} = (g_j \wedge g_i)\cdot R(g_k \wedge g_l) \tag{18}$$

and the covariant components of the Ricci tensor are given by

$$R_{ij} = (g_i \wedge g^k)\cdot R(g_k \wedge g_l) \tag{19}$$

where a summation on the index k is assumed. All these relations can be written in a coordinate base as well as in a noncoordinate one.

5. COMPUTING FACILITIES

Here we give a translation of the equations of Section 4 into notation understood by GEOCALC. Let us recall that a point stands for the inner product, a tilde (~) stands for outer product, a vertical bar stands for geometric product, and the {,} stand for the cross product (Tombal and Moussiaux, 1986). When products involve the differential operators **d** and ⍁ the previous symbols must be "doubled."

Additions and differences are performed by the operators ++ and −−.

Formula (17) becomes

om[*i*]:= gad[*i*] ~ (grad||*h*[*i*])

The curvature tensor (15) is given by

r[*i*, *j*]:= dom[*i*, *j*]−−dom[*j*, *i*]++crom[*i*, *j*]

where

dom[*i*, *j*]:= msum((*d*[*i*]||(hinv[*k*]|(*d*[*k*]||*h*[*j*])))
|(gad[*j*] ~ gau[*k*]), *k*, %indx)

and where

crom[*i*, *j*]:= {om[*i*], om[*j*]} (20)

The components of the Riemann tensor are computed in the following way:

rie[*i*, *j*, *k*, *l*]:= (gd[*j*] ~ gd[*i*]).*r*[*k*, *l*]

In these formulas gd[*j*] stands for g_j, gad[*i*] stands for γ_i. Comparison with the relations of Section 4 gives readily the meaning of the other symbols.

6. DISCUSSION OF THE RESULTS

First of all, our results are in full accordance with those obtained by the program CTENSR (a package for tensor manipulations written in MACSYMA), but there are some results that do not agree with those of Hestenes (1986).

Computation of (20), which corresponds to equations (36) in Hestenes (1986), gives the following results:

$$\mathrm{crom}_{0,1} = @(0, [\ ])$$

$$\mathrm{crom}_{0,2} = @\left(\frac{\% e^{\mathrm{phi}}\, d\mathrm{phi}/dx\%_1}{(\% e^{\mathrm{lam}})^2}, [0, 2]\right)$$

$$\mathrm{crom}_{0,3} = @\left(\frac{\sin(x\%_2)\ \% e^{\mathrm{phi}}\, d\mathrm{phi}/dx\%_1}{(\% e^{\mathrm{lam}})^2}, [0, 3]\right)$$

$$\mathrm{crom}_{1,2} = @\left(\frac{d\mathrm{lam}/dx\%_0}{\% e^{\mathrm{phi}}}, [0, 2]\right)$$

$$\mathrm{crom}_{1,3} = @\left(\frac{\sin(x\%_2)\ d\mathrm{lam}/dx\%_0}{\% e^{\mathrm{phi}}}, [0, 3]\right)$$

$$\mathrm{crom}_{2,3} = \left[@\left(-\frac{\cos(x\%_2)}{\% e^{\mathrm{lam}}}, [1, 3]\right), @\left(\frac{\sin(x\%_2)}{(\% e^{\mathrm{lam}})^2}, [2, 3]\right)\right]$$

We see that the two first equations are in agreement with those of Hestenes, but the sign of the last three are different.

On the other hand, our results are in full agreement with equations (37) in Hestenes (1986) provided that we use the relation (16) or the general relation (22) in Hestenes (1986) instead of (31) in Hestenes (1986).

Note that in addition to (37) in Hestenes (1986) we find two other nonvanishing equations $d_t\omega_t$ and $d_r\omega_r$:

$$\mathrm{dom}_{0,0} = @\Bigg(-\% e^{\mathrm{phi}-\mathrm{lam}} \left[\left(\frac{d\mathrm{phi}}{dx\%_0} - \frac{d\mathrm{lam}}{dx\%_0}\right)\frac{d\mathrm{phi}}{dx\%_1} + \frac{d^2\mathrm{phi}}{dx\%_0\, dx\%_1}\right], [0, 1]\Bigg)$$

$$\mathrm{dom}_{1,1} = @\Bigg(\% \ e^{\mathrm{lam}-\mathrm{phi}} \left(\frac{d\mathrm{lam}}{dx\%_0}\frac{d\mathrm{phi}}{dx\%_1} - \frac{d\mathrm{lam}}{dx\%_0}\frac{d\mathrm{lam}}{dx\%_1} - \frac{d^2\mathrm{lam}}{dx\%_0\, dx\%_1}\right), [0, 1]\Bigg).$$

Consequently, the relations (38) in Hestenes (1986) giving the components of the curvature tensor (15) are also different and $\omega_{t\theta}$ and $\omega_{t\phi}$ must be replaced by

$$\omega_{t\theta} = e^{\phi} r\gamma_\theta \wedge (\gamma_r \lambda_{,t} r^{-1} e^{-\phi-\lambda} - \gamma_t r^{-1} \phi_{,r} e^{-2\lambda})$$

$$\omega_{t\phi} = e^{\phi} \sin\theta\, \gamma_\phi \wedge (\gamma_r \lambda_{,t} e^{-\phi-\lambda} - \gamma_t \phi_{,r} e^{-2\lambda})$$

If we want to obtain the correct components of the Riemann and Ricci tensors, the expressions (39) in Henstenes (1986) must be computed using the quantities modified as explained above.

7. CONCLUSIONS

Despite inevitable computational errors, the "fiducial frame" method proposed by Hestenes (1986) is a powerful means of investigation in general relativity because it combines the facilities of Cartan's method with the systematic character of usual tensorial analysis. From these computational points of view, Hestenes' method has the great advantage of allowing "punctual" calculations, that is, any component of a given tensor may be computed independently.

This method is a particular application of geometric calculus (Hestenes and Sobcszyk, 1984) based on Clifford algebra, whose field of investigation extends largely beyond general relativity.

In a previous paper (Tombal and Moussiaux, 1986) we showed how we implemented geometric calculus in a package called GEOCALC written in MACSYMA. In this paper we showed an application of the package GEOCALC to the "fiducial frame" method. Other metrics (Robertson-Walker, for instance) have also been investigated and the results provided by GEOCAL are in full accordance with those obtained using different computational methods.

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